# Maclaurin and Taylor Series

## What are Maclaurin and Taylor Series?

Maclaurin and Taylor series are two types of power series that can be made from functions of real numbers. We will now look at how these series are constructed for a function:

**Definition.** *The Maclaurin Series of a function f is given by:* $MS(f(x)) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$

**Definition.** *The Taylor Series of a function f is given by:* $TS(f(x), a) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n$

As you can see Maclaurin series are just a special type of Taylor series, therefore all theorems for Taylor series also apply to Maclaurin series (but not the other way around). All of our "usual" functions have Taylor and Maclaurin series, but it is not always true that the resulting series converges to the original functions. The next theorem gives a sufficient condition for convergence.

**Theorem.** *If a function f has a power series representation (expansion) at a, that is if* $f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$ *where* $|x-a| < R$*, then its coefficients are given by the formula* $c_n = \frac{f^{(n)}(a)}{n!}$

So as a result:
**If** f has a power series representation at a, **and** $|x-a| < R$
**then** $f(x) = TS(f(x), a)$

## Table of Common Maclaurin Series

| *f(x)* | *MS(f(x))* | *MS(f(x))* (expanded) | Radius of Convergence |
|---|---|---|---|
| $\frac{1}{1-x}$ | $\sum_{n=0}^{\infty} x^n$ | $1 + x + x^2 + x^3 + \dots$ | $R = 1$ |
| $e^x$ | $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ | $1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$ | $R = \infty$ |
| $\sin x$ | $\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$ | $x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$ | $R = \infty$ |
| $\cos x$ | $\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$ | $1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$ | $R = \infty$ |
| $\tan^{-1} x$ | $\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$ | $x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots$ | $R = 1$ |
| $(1+x)^k$ | $\sum_{n=0}^{\infty} \binom{k}{n} x^n$ | $1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \dots$ | $R = 1$ |