Integration Techniques

Substitution

Substitution is often used in rational functions where the numerator is the derivative of the denominator.

Example 1

$$\int \frac{2x}{(x^2-2)^7}dx$$

Let $u = x^2 - 2 \implies du = 2xdx$
Substituting we get $\int \frac{2x}{(x^2-2)^7 dx} = \int \frac{1}{u^7}du = \int u^{-7}du == \frac{u^{-6}}{-6} + C = \frac{(x^2-2)^{-6}}{-6} + C = \frac{-1}{6(x^2-2)^6} + C$

Another common use of substitution is for integrals of trigonometric functions, which will be discussed in another section. This integral will show one of the common strategies involving substitution.

Example 2

$$\int (x^2\ \sqrt{x+1})dx$$

Let $u = \sqrt{x+1} \implies u^2 = x+1 \implies u^2 - 1 = x \implies 2udu = dx$
Substituting we get $\int (u^2-1)u \cdot 2u \cdot du = \int (2u^4 - 2u^2)du = \frac{2u^5}{5} - \frac{2u^3}{3} + C = \frac{2(\sqrt{x+1})^5}{5} - \frac{2(\sqrt{x+1})^3}{3} + C$

Trigonometric Substitution

Case 1: $\sqrt{a^2 - x^2}$ set $x = a\sin\theta$

Case 2: $\sqrt{a^2 + x^2}$ set $x = a\tan\theta$

Case 3: $\sqrt{x^2 - a^2}$ set $x = a\sec\theta$

Example 3 $\int \frac{1}{\sqrt{x^2-6x+5}}dx = \int \frac{1}{\sqrt{x^2 - 2\cdot 3\cdot x + 5 + 9 - 9}}dx$ *(by Completing the Square)*

$$= \int \frac{1}{\sqrt{(x-3)^2-4}}dx$$

let $x - 3 = 2\sec\theta \implies x = 3 + 2\sec\theta \implies dx = 2\sec\theta\tan\theta$

$$\int \frac{2\sec\theta\tan\theta}{\sqrt{4\sec^2\theta - 4}}d\theta = \int \frac{2\sec\theta\tan\theta}{\sqrt{4\sec^2\theta - 4}}d\theta = \int \frac{2\sec\theta\tan\theta}{2\sqrt{\sec^2\theta - 1}}d\theta = \int \frac{\sec\theta\tan\theta}{\tan\theta}d\theta = \int \sec\theta d\theta = \ln|\sec\theta + \tan\theta| + C$$

$\tan\theta = \sqrt{\sec^2\theta - 1} = \sqrt{\left(\frac{x-3}{2}\right)^2 - 1}$
Substituting

$$\ln\left|\frac{x-3}{2} + \sqrt{\left(\frac{x-3}{2}\right)^2 - 1}\right| + C$$

Integration By Parts

Integration By Parts: $\int udv = uv - \int vdu$
OR $\int uv = u \int v - \int (\int v) du$

Let's solve the following:

Example 4

$$\int 6\tan^{-1}\left(\frac{8}{x}\right) dx = 6 \int 1 \cdot \tan^{-1}\left(\frac{8}{x}\right) dx$$

$$\int 6\tan^{-1}(\frac{8}{x})dx = 6x\tan^{-1}\left(\frac{8}{x}\right) - 6\int x \cdot \frac{1}{1+\frac{8}{x^2}}dx = 6x\tan^{-1}\left(\frac{8}{x}\right) - 6\int x \cdot \frac{x^2}{x^2+8}dx = 6x\tan^{-1}\left(\frac{8}{x}\right) - 6I_1$$

let $u = x^2$ *in* $I_1 \implies dx = \frac{u}{2}du$

$$I_1 = \frac{1}{2}\int \frac{u}{u+8}du = \frac{1}{2}\int \frac{u+8-8}{u+8}du = \frac{1}{2}\int \frac{1}{u+8}du - \frac{1}{2}\int \frac{8}{u+8}du = \frac{1}{2}\ln|u+8| + 4\ln|u+8| + C$$

$$= \frac{9}{2}\ln|u+8| = \frac{9}{2}\ln|x^2+8| + C$$

Partial Fraction

$P(x)$ is a polynomial with degree less than or equal to $Q(x)$ in $\dfrac{P(x)}{Q(x)}$.

Case 1: $Q(x) = (ax+b)^k$
$$\frac{P(x)}{Q(x)} = \frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \cdots + \frac{A_k}{(ax+b)^k}$$

Case 2: $Q(x) = (ax^2+bx+c)^k$
$$\frac{P(x)}{Q(x)} = \frac{A_1x+B_1}{(ax^2+bx+c)} + \frac{A_2x+B_2}{(ax^2+bx+c)^2} + \cdots + \frac{A_kx+B_k}{(ax^2+bx+c)^k}$$

You can use these cases in the same question like below.

Example 5

$$\int \frac{\left(4x^3+12x^2+11x+6\right)}{(x+1)^2\left(x^2+x+1\right)}dx = \int \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{Cx+D}{x^2+x+1}dx$$

$$\left(4x^3+12x^2+11x+6\right) = (Cx+D)(x+1)^2 + A(x+1)\left(x^2+x+1\right) + B\left(x^2+x+1\right)$$

$$A = 2, B = 3, C = 2, D = 1$$

$$\implies \int \frac{\left(4x^3+12x^2+11x+6\right)}{(x+1)^2\left(x^2+x+1\right)}dx = \int \frac{2}{x+1} + \frac{3}{(x+1)^2} + \frac{2x+1}{x^2+x+1}dx = 2\ln|x+1| - \frac{3}{x+1} + \ln\left|x^2+x+1\right| + C$$