First Order ODE's

1. **Seperable Equations**

$$f(y)y' = g(x)$$

Example 1.

$$y' = x^2(4y^2+1) \implies \frac{1}{4y^2+1}y' = x^2 \implies \frac{1}{2}\tan^{-1}(2y) = \frac{x^3}{3}+C \implies \tan^{-1}(2y) = \frac{2x^3}{3}+C \implies y = \frac{1}{2}\tan\left(\frac{2x^3}{3}+C\right)$$

2. **Homogenous First Order DEs** The method:

$$y' + p(x)y = 0 \implies \frac{y'}{y} = -p(x) \implies \int \frac{y'}{y} = -\int p(x) \implies ln|y| = -P(x) + ln|c|$$

Here $\ln|c|$ is equivalent to c because any number is some number's natural log

$$\implies y = e^{-P(x)} \cdot C$$

3. **Variation Of Parameters OR Integrating Factor**

$$y' + p(x)y = f(x)$$

If you know the solution to the DE $y' + p(x)y = 0$ say y_1 then solution to about DE would be of the form $y = uy_1$ for some variable u. As, $y = uy_1 \implies y' = u'y_1 + uy_1'$. Substitute in the original DE and you will get $u' = \frac{f(x)}{y_1}$. Note: substitute $y_1' + p(x)y_1 = 0$. **comment You don't need to memorize these formulas but I will make use of them to keep the handout brief.**

Example 2.

$$\therefore \frac{u'}{u^{-1}} = \frac{1}{(1+x^2)^2 y_1^{-1-1}} \implies uu' = \frac{1}{(1+x^2)^2} \cdot (1+x^2)^2$$

$$\implies \int uu' = \int 1 \implies \frac{u^2}{2} = x \implies u = \sqrt{2x}$$

$$Hence,\ y = \frac{\sqrt{2x}}{1+x^2}$$

4. **Bernoulli Equations**

$$y' + p(x)y = f(x)y^r$$

This can be solved using Variation of Parameters. You should be able to get the relation $\frac{u'}{u^r} = f(x)y_1^{r-1}$

$$(1+x^2)y' + 2xy = \frac{1}{(1+x^2)y} \implies y' + \frac{2x}{(1+x^2)}y = \frac{1}{(1+x^2)^2 y}$$

Solving the equation $y' + \frac{2x}{(1+x^2)}y = 0$

$$\implies y_1 = e^{-\int \frac{2x}{(1+x^2)}dx} \implies y_1 = e^{-\ln|1+x^2|} = \frac{1}{1+x^2}$$

$$\therefore \frac{u'}{u^{-1}} = \frac{1}{(1+x^2)^2 y_1^{-1-1}} \implies uu' = \frac{1}{(1+x^2)^2} \cdot (1+x^2)^2$$

$$\implies \int uu' = \int 1 \implies \frac{u^2}{2} = x \implies u = \sqrt{2x}$$

Hence, $y = \frac{\sqrt{2x}}{1+x^2}$

5. **Exact Equations** If a DE of the form : $M(x,y)dx + N(x,y)dy = 0$ has $M_y = N_x$ where G_x represents the partial derivative of a function G with respect x then, the equation is called exact. If this is the form then we can find $F(x,y)$ such that $F_x = M(x,y)$ and $F_y = N(x,y)$.

Example 3.

$$(3y\cos(x) + 4xe^x + 2x^2e^x)dx + (3\sin(x) + 3)dy = 0$$
$$M = 3y\cos(x) + 4xe^x + 2x^2e^x \implies M_y = 3\cos(x)$$
$$N = 3\sin(x) + 3 \implies N_y = 3\cos(x)$$

Hence the equation is exact. $F_x = 3y\cos(x) + 4xe^x + 2x^2e^x$ *is difficult to integrate, so choose,* $F_y = 3\sin(x) + 3$ *which is easy.*

$$\int F_y dy = \int (3\sin(x) + 3)dy$$
$$F = 3y\sin(x) + 3y + h(x) + C$$

Now need to find $h(x)$.

$$F_x = 3y\cos(x) + h'(x) = 3y\cos(x) + 4xe^x + 2x^2e^x$$
$$h'(x) = 4xe^x + 2x^2e^x$$

This is a special integral of the form $\int e^x(f(x) + f'(x)) = e^x f(x)$

$$h(x) = 2x^2e^x$$

$\therefore F(x,y) = 3y\sin(x) + 3y + 2x^2e^x + C$ *which is the solution to the DE.*

6. **Almost Exact Equations** If $M_y \neq N_x$, there is still hope. We can use integrating factor μ such that, if $q(x) = \dfrac{M_y - N_x}{N}$ is independent of y or if $p(y) = \dfrac{N_x - M_y}{M}$ is independent of x then $\mu(x) = \pm e^{\int q(x)}$ if the first condition is met or $\mu(y) = \pm e^{\int p(y)}$ if the second condition is met. If both are met $\mu(x,y) = q(x) \cdot p(y)$.

Example 4.

$$(27xy^2 + 8y^3)dx + (18x^2y + 12xy^2)dy = 0$$

The fact that equation is not exact could be easily verified.

$$M_y = 54xy + 24y^2 \quad \& \quad N_x = 36xy + 12y^2$$

situation for $\frac{N_x - M_y}{M}$ *is left as an exercise.*

$$q(x) = \frac{M_y - N_x}{N}$$
$$q(x) = \frac{18xy + 12y^2}{18x^2y + 12xy^2} = \frac{y(18x + 12y)}{xy(18x + 12y)} = \frac{1}{x}$$
$$\mu(x) = \pm e^{\int \frac{1}{x}} = \pm x$$

Multiply the DE by x.

$$(27x^2y^2 + 8xy^3)dx + (18x^3y + 12x^2y^2)dy = 0$$

This DE is exact and can be solved as shown in the above section.